Conditional stability of shock waves—a criterion for detonation

G. R. Fowles

Department of Physics, Washington State University, Pullman, Washington 99163

(Received 7 July 1975)

The stability of plane shock waves is treated by examining the amplitudes of acoustic waves reflected from shock fronts, and by methods of irreversible thermodynamics. Both approaches yield the same conditions for stability, $-1 \le j^2(dV/dP)_H \le 1$, where j^2 is the negative slope of the Rayleigh line, and the derivative is taken along the Hugoniot P-V curve. The thermodynamic method indicates that instabilities are associated either with local maxima in the entropy, or shock velocity; or with local minima in the reduced internal energy, or particle velocity, along the Hugoniot curve. It is proposed that the latter case corresponds to detonation with the detonation state given by the particle velocity minimum.

I. INTRODUCTION

Earlier studies of the stability of shock waves have established the existence of two limits outside which a shock splits spontaneously into two waves traveling in the same or in opposite directions. Bethe first derived sufficient conditions for plane shocks to be stable against such breakup.[1] Later studies by D'yakov,[2] and by Erpenbeck,[3] based on analysis of the stability with respect to two-dimensional perturbations also established two bounds; these were shown by Gardner to be equivalent to Bethe's criteria for plane shocks.[4,5]

In this paper we consider a region within the above limits in which a shock is evidently potentially unstable for other reasons. We show that in this region small amplitude acoustic waves incident on the shock front from the compressed region behind the front undergo amplification upon reflection at the front. This can lead to an oscillatory type of instability proposed earlier,[6] although it is not clear from this treatment that instability necessarily occurs when the amplification criterion is satisfied.

We have also approached the stability problem from the point of view of irreversible thermodynamics and show, based on a plausible hypothesis, that in the region under consideration a shock is thermodynamically unstable; whether or not instability actually occurs depends on the magnitude of perturbations. The acoustic wave approach and the thermodynamic approach thus exhibit a nice correspondence.

Technical interest in the shock stability problem derives from applications in which it is desired to relate wave propagation behavior to properties of the transmitting medium. In solids, for example, polymorphic phase changes and yielding at the elastic limit may lead to splitting of a single shock into two shocks traveling in the same direction. In reactive media, self-sustaining waves or detonation waves, may form under conditions that are not well understood.

The problem is also of exceptional theoretical interest because of the existence of several apparently distinct methods of approach, as has been pointed out by Woods.[7] The theory of irreversible thermodynamics is well known to be underdeveloped, and it may be hoped that new insight into the theory will result from application of various methods to a relatively simple problem such as that of plane shock waves.

The thermodynamic method employed here invokes no new principles but requires the recognition that the approach to equilibrium of two systems initially out of equilibrium is characterized by nonnegative entropy production in each system. This can be expressed, at least for adiabatic, viscous flow, by an upper as well as a lower bound to the entropy production rate. Still another statement is that the reduced internal energy (defined later) is minimized *and* the entropy is maximized in equilibrium. These latter statements are not, in general, equivalent; one does not imply the other.

The thermodynamic method predicts a new criterion for detonation that is quite different from the Chapman–Jouguet Theory. We postulate this criterion in Sec. V.

In Sec. II we display the jump conditions and several definitions and transformations that are useful. Section III is a summary of the conclusions of the Bethe–D'yakov theory. The interaction of acoustic waves with the shock front is considered in Sec. IV and the thermodynamic approach is presented in Sec. V.

II. JUMP CONDITIONS

The well-known Rankine–Hugoniot jump conditions applicable to plane shocks with steady profile or to discontinuous jumps can be written,[8]

$$u - u_0 = \rho_0(U - u_0)(V_0 - V) , \quad (1)$$

$$\sigma - P_0 = \rho_0(U - u_0)(u - u_0) , \quad (2)$$

$$E - E_0 = \tfrac{1}{2}(\sigma + P_0)(V_0 - V) . \quad (3)$$

These equations express conservation of mass, momentum, and energy, respectively. Mass velocity is denoted by u, shock velocity by U, specific volume by $V = \rho^{-1}$, normal stress in the direction of propagation by σ (measured positive in compression), and specific internal energy by E. Subscripts 0 refer to the undisturbed state ahead of the shock, assumed to be a thermodynamic equilibrium state. The mechanical conditions, Eqs. (1) and (2) require no assumption about thermodynamic equilibrium and apply throughout the shock transition region; hence, the use of σ to denote stress rather than P which is used to denote the pressure of thermodynamic equilibrium states. Equation (3) is valid whenever no other sources of energy besides mechanical energy are assumed.

Since a shock is an adiabatic process, Eq. (3) applies to equilibrium end states; it only applies to the shock transition region, however, when heat conduction and radiation in that region can be neglected. Equation (3) is termed the Hugoniot relation and, for given (P_0, V_0, E_0), defines a surface,

$$\sigma=\sigma(V,E;P_0,V_0,E_0) \quad (V\neq V_0),$$

that represents the locus of states achievable by a shock transition in *any* medium.

For the description of shocks in a specific medium, Eqs. (1)–(3) are supplemented by the equilibrium equation of state of the medium in the form

$$P=P(V,E;E_0). \tag{4}$$

The simultaneous solution of Eqs. (3) and (4), with $\sigma=P$, yields a curve $P(V)$ termed the "Hugoniot equation of state," or sometimes the "R-H curve."

We define several useful quantities

$$j\equiv\rho_0(U-u_0), \tag{5}$$

whence, from Eqs. (1) and (2),

$$j^2\equiv(\sigma-P_0)/(V_0-V). \tag{6}$$

Also,

$$M\equiv|(U-u)/c| \tag{7}$$

and

$$c^2\equiv\left(\frac{\partial P}{\partial\rho}\right)_s=-V^2\left(\frac{\partial P}{\partial V}\right)_s. \tag{8}$$

The quantity j is the mass flux through the shock front and is positive when the shock velocity exceeds the initial mass velocity u_0. Its square j^2 is also equal to the negative slope of the Rayleigh line joining the end states. The quantity M is the local Mach number of the shock with respect to the medium, and c is the local sound speed in spatial coordinates. The subscript s denotes the isentropic derivative.

Several combinations of these relations yield useful transformations. Thus, combining Eqs. (1), (2), and (6),

$$(u-u_0)^2=(\sigma-P_0)(V_0-V)=j^2(V_0-V)^2. \tag{9}$$

This can be differentiated to give

$$2(u-u_0)du=-(\sigma-P_0)dV+(V_0-V)d\sigma,$$

or, using Eq. (9),

$$j\left(\frac{du}{dP}\right)_H=\pm\frac{1}{2}\left[1-j^2\left(\frac{dV}{dP}\right)_H\right], \tag{10}$$

where the subscript H denotes differentiation along the Hugoniot curve.

For definiteness we consider only compressive shocks traveling in the positive direction, so that,

$$j>0;\quad V<V_0;\quad \text{and}\quad u>u_0.$$

As a result of this assumption we retain only the positive sign in Eq. (10).

An alternate expression for Eq. (7) can be derived, using Eqs. (5), (8), and (9),

$$M^2=\left|\frac{U-u}{c}\right|^2=\frac{[V_0j-(V_0-V)j]^2}{-V^2(\partial P/\partial V)_s}$$

$$=-j^2\left(\frac{\partial V}{\partial P}\right)_s. \tag{11}$$

For small amplitude acoustic waves we make use of the characteristic equations and associated compatibility conditions[8]

$$C\pm:\ dx/dt=u\pm c \tag{12a}$$

and

$$\Gamma\mp,\ \text{or}\ S\pm:\ dP/du=\pm\rho c. \tag{12b}$$

The upper sign of Eq. (12b) holds *across* forward-facing waves, specified by the positive sign of Eq. (12a). Thus, Γ^+ is valid *on* the characteristic path C^+, and Γ^- holds on C^-. For acoustic waves the flow is assumed to be isentropic, and we therefore adopt the obvious notation for these waves

$$\left(\frac{du}{dP}\right)_s=\pm(V/c).$$

Combining this with Eqs. (6) and (11) gives,

$$\left(\frac{du}{dP}\right)_s=\pm\left(-\frac{\partial V}{\partial P}\right)_s^{1/2}$$

$$=\pm(M/j). \tag{13}$$

Still another useful relation can be obtained by writing the slope of the Hugoniot curve as a directional derivative,

$$\left(\frac{dP}{dV}\right)_H=\left(\frac{\partial P}{\partial V}\right)_E+\left(\frac{\partial P}{\partial E}\right)_V\left(\frac{dE}{dV}\right)_H,$$

and employing Eq. (3), which differentiated is, with $\sigma=P$,

$$\left(\frac{dE}{dV}\right)_H=-\tfrac{1}{2}(P+P_0)+\tfrac{1}{2}(V_0-V)\left(\frac{dP}{dV}\right)_H.$$

Thus,

$$\left(\frac{\partial P}{\partial V}\right)_E=\left(\frac{dP}{dV}\right)_H+\frac{1}{2}\left(\frac{\partial P}{\partial E}\right)_V\left[P+P_0-(V_0-V)\left(\frac{dP}{dV}\right)_H\right].$$

However, on the equilibrium surface,

$$\left(\frac{\partial P}{\partial V}\right)_E=\left(\frac{\partial P}{\partial V}\right)_s-\left(\frac{\partial P}{\partial E}\right)_V\left(\frac{\partial E}{\partial V}\right)_s=\left(\frac{\partial P}{\partial V}\right)_s+P\left(\frac{\partial P}{\partial E}\right)_V.$$

The Grüneisen parameter is

$$\Gamma=V(\partial P/\partial E)_V$$

Hence, equating the two expressions for $(\partial P/\partial V)_E$,

$$\left(\frac{dV}{dP}\right)_H=\frac{1-(\Gamma/2V)(V_0-V)}{(\partial P/\partial V)_s+(\Gamma/2V)(P-P_0)}. \tag{14}$$

This can be simplified by the substitution

$$a=(\Gamma/2V)(V_0-V), \tag{15}$$

together with Eq. (11). We get

$$j^2(dV/dP)_H=M^2(a-1)/(1-M^2a). \tag{16}$$

A graph of this equation is shown in Fig. 1.